Designing an Early Number Sequence for Teaching

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This paper discusses the design modifications of a sequence of specified learning goals for teaching early number, which characterise an instructional resource developed in a design research collaboration with teachers in Mexican preschools. Highlighted are the types of research involvement that appear to be necessary in designing resources for teaching meaningful mathematics at scale. We argue that, and illustrate how, such involvements need to venture beyond addressing the problems of learning, into the territory of problems of teaching.

In this paper, and in our recent work, we grapple with the relation between researcher-produced instructional design resources and mathematics teaching. We take the perspective that teachers ultimately shape how mathematical activities play out (Pepin, 2018), and what forms of mathematical reasoning become characteristic of conversations in their classrooms. We then see the job of design researchers to be that of re-sourcing teachers (Pepin et al., 2013) for the demanding, intellectual task at their hand.

From a perspective of insiders to a design research tradition in mathematics education (Cobb et al., 2003), we draw out a distinction between material-and-conceptual-resources that are a product of classroom design experiments (CDEs) focused on student mathematical learning (e.g., Cobb, Gravemeijer, et al., 1997) and those that could be viable for re-sourcing teaching at scale. We seek to address the research question of what we need to understand when designing resources for teachers' use.

To open this question, we share our failures and successes in designing for teaching in a context of early number in Mexican preschools. We exemplify our learning that resulted from taking problems of teaching seriously across cycles of design. Specifically, we discuss iterations of the goals for student learning we shared with teachers, and how the teachers' interpretations of and responses to those goals led to their subsequent, more viable redesign.

Our design research included (a) an adaptation of an instructional theory in early number (Cobb, Gravemeijer, et al., 1997) for the learning of younger children; (b) an intensive collaboration with a single kindergarten teacher Jesica, the third author, conducting a CDE (Peña, 2018; Peña et al., 2018); (c) a collaboration, orchestrated by Jesica, with a group of teachers; and (d) the facilitation of intensive online workshops for groups of preschool teachers, and of an online community that formed around the designed resource.

Designing for Teaching

When conducting CDEs, researchers design, test, and modify the means of supporting the learning they strive to study, because to be able to study such learning, they must be able to first generate it in the classroom (Cobb et al., 2003). For instance, guided by an adaptation of the theory of Realistic Mathematics Education (Cobb et al., 2008), Cobb and colleagues pursued questions about collective mathematical developments that *can be generated* in classroom communities (even if such developments currently do not take place in any classrooms), and what forms of reasoning need to be sequentially supported in a classroom for such developments to become a reality. It may appear that both the formulated learning goals

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and the means by which these were achieved in CDEs should be shareable with mathematics teachers rather seamlessly. However, this does not appear to be the case.

Teachers have been shown to be highly capable of developing the complex and demanding teaching practices that require that they analyse, adapt, test and refine, in their own classrooms, the resources that were developed and proved effective in CDE classrooms (e.g., Visnovska & Cobb, 2019). However, the structural and institutional support conditions that are available to design research teams are rarely in place for teachers. As a result, while teachers may face problems in their students' learning, they face additional, immediate and pervasive *problems of teaching*, related to affordances and constraints of a mandated curriculum, assessments, or how their teaching is locally organised (Lampert, 2001). The resources produced in CDEs with a focus on student learning (e.g., Cobb, Gravemeijer, et al., 1997) are typically not designed to address these additional problems. In this paper, we discuss cycles of design research through which we strove to address the problems of teaching via resource design to make the use of novel resources viable for teachers.

It is important to clarify that when discussing a designed resource, we do not refer to a collection of classroom tasks that are given to the teachers to implement. The resource could instead be better imagined as a system of assumptions and forms of mathematical and pedagogical reasoning that allow teachers to independently pursue, in their classroom, a sequence of specified learning goals, while assessing the completion of these goals and deciding when to adapt/design additional classroom activities and when to *move on* to the next goal. The paramount goal of the designed resource is that teachers experience agency over the collective mathematical learning and education that takes place in their classroom and perceive their job as manageable.

Background and Method

The collaboration with Jesica started when she enrolled in a Masters degree program, with the first author as her advisor. Jesica taught in a public preschool attended by children from very-low-income families. In Mexico, *preschools* serve children in three year levels, prior to their transition to Year 1 of a primary school. At the start of preschool, the youngest children are 2 years and 8 months old. During her six years of teaching, Jesica was mostly in charge of Year 3 preschool classrooms, comparable to a Foundation Year in Australia.

For her Masters degree, Jesica conducted a CDE on early number with a classroom of 22 Year 3 preschool children, findings of which she reported in Peña (2018). Her study was a part of a dual design experiment (Gravemeijer & van Eerde, 2009), in which the first two authors collected and analysed data on Jesica's learning and resource use. Jesica later took a teaching job in a larger preschool, where her teaching practice attracted attention of colleagues, and led, over time, to a broader exploration of the teaching resource, when additional teachers trialled it in their classrooms. At this time, Jesica acted as a broker between the communities of the researchers (the 4 authors) and of teachers in her school and school-cluster. Efforts at supporting the group of teachers led to an emergence of a sizeable online community, and a provision of intensive teacher professional development events.

The method that allows us to make claims about teachers' uses of the designed resource is an adaptation of the method used to study student learning in CDEs (e.g., Cobb & Whitenack, 1996). Researchers collect data to document (a) research conjectures on what teachers will do and why doing so would be reasonable within their context (e.g., how they would use a specific aspect of the resource and why), (b) what teachers actually did, and (c) how teachers explained and justified their teaching decisions. During the ongoing analysis, researchers aim to understand where their initial conjectures about teachers' resource use "went wrong", adjust their understanding of how teachers reasoned with the resource in the specific context of their schools, adjust conjectures about teachers' future activity and, if needed, redesign the resource to better support teachers' decision-making process. A retrospective analysis is conducted upon completion of a study cycle to interpret, with hindsight, which resource modifications made the difference to the teachers and how the resource could be further modified prior to the next cycle of trials.

Both during the dual design experiment, and in her role as a broker, Jesica was the key person responsible for the data collection—on her own and others' attempts at the resource use, and on the reasoning that underpinned these uses. Multiple corrective processes are in place during design research studies to correct for possible inaccuracies in data collection and interpretation, the most powerful of them being the reality check. Had the data not represented the resource uses with integrity, the design modifications developed on their basis would have had no chance of better supporting teachers' reasoning with, or use of, the resource. Indeed, even modifications based on reliable data often fail to succeed, as designing for a real difference requires considerable cumulative learning on the researchers' part. A modification that leads to a productive change in teacher resource use, retrospectively, verifies the viability of the model of teacher reasoning that gave rise to it. The researcher's (in this case Jesica's) results and failures in CDE teaching, through which the resource was initially developed or tested, provide an important interpretive tool in making sense of other teachers' classroom experiences.

The Cycles of Designing for Teaching

At the start of the project, the research team shared a concern for the limitations in early number teaching in Mexican preschools. The sources of these concerns were, however, different. The members who were trained researchers were aware of the general poor performance of Mexican students on national and international standardised assessments and of the findings indicating that underperformance in preschool is predictive of subsequent mathematical performance (e.g., Jordan et al., 2009).

Jesica's concerns came from her experience as a preschool teacher, where she had seen only very few of her Year 3 students ever meeting the learning goals specified in the Mexican curriculum. This was the case despite her commitment to her students' learning, her interest in mathematics education, her commitment to taking professional development courses in the subject, and the attention she paid to the recommendations published by the Mexican Ministry of Education on teaching mathematics in preschool, which she tried to faithfully apply.

The initial project involved supporting Jesica in conducting a CDE, in which she tested the viability of the *Patterns and Partitioning* (P&P; Cobb, Gravemeijer, et al., 1997) instructional sequence. The sequence was designed to support the collective development of early number ideas by providing instructional opportunities for students to reason about patterns and partitions of collections of up to ten items. For example, students are supported to develop familiarity with pairs of numbers that add up to five, by finding the different ways in which five monkeys could be in two trees (Cobb, Boufi et al., 1997).

Jesica's CDE project was justifiable as the P&P sequence had been tested with older, US students (Year 1), with a significantly different socio-economic background. Prior to engaging in the CDE, these students already had a command of the verbal number sequence up to 10, recognised written numerals up to 10, and had an understanding of cardinality when counting small collections of objects.

Cycle 1: Adapting the Instructional Sequence to Teach Younger Students

At the start of the CDE led and taught by Jesica, we assessed the pre-schoolers. The results indicated that most of them were not ready to productively engage in the initial activities of the P&P sequence, as they yet had to develop elementary number abilities. Some children were only successful with the word number sequence up to three and could correctly identify the names of only one or two single digit numerals. Activities like *Monkeys* (above) would not be within their reach. This led to the reformulation of the sequence (see Table 1).

Table 1
The P&P Instructional Sequence Implemented in Cycle 1

Phase	Overarching teaching goal	Specific learning goals
1	Support the development of the essential number understandings up to five	Master the number word sequence Enumerate with one-to-one correspondence Use fingers to represent numbers Identify the names of written numerals
2	Support students' reasoning about patterns and partitioning with numbers up to five	Reason about (and subitise) spatial patterns Reason about (and subitise) finger patterns Reason about number partitions in the 10-frame Subitise and reason about spatial patterns in the 10-frame Reason about arithmetic problems
3	Support the development of the essential number understandings up to ten	Master the number word sequence Enumerate with one-to-one correspondence Use fingers to represent numbers Identify the names of written numerals
4	Support students' reasoning about patterns and partitioning with numbers up to ten	Reason about (and subitise) finger patterns Subitise and reason about spatial patterns in the 10-frame Reason about number partitions in the arithmetic rack Reason about arithmetic problems

Prior to introducing the original P&P sequence activities, we included a phase, in which we intended to support the children to develop essential number understandings with numbers up to five (Phase 1). We also formulated a general strategy to first support the reasoning about P&P with numbers up to five, and only then proceed to working with larger numbers (we separated Phases 2 and 4 and added an explicit focus on essential understandings of larger numbers in Phase 3). This distinction was not key in the initial sequence, given the more advanced starting point of the US students.

The CDE consisted of 21 instructional sessions that were implemented over a 5-month period. Our analysis indicated that the reformulated instructional sequence was viable. Peña (2018) reported that the learning goals of Phase 1 were met after five sessions, in which the teacher supported collective engagement in activities of repeated counting with words and symbols. The activities included singing number songs, playing number-word games and board games. To encourage the repeated counting of small collections, the students were asked to help the teacher's friend, who owned a candy factory, to find out how she could sell the candies without having to count them one by one. Using a large supply of Unifix cubes, students created rods of *candy packs* of the same size (e.g., 4 cubes).

The learning goals of Phase 2 were met after six more instructional sessions, where each of the five learning goals were accomplished (Peña, 2018). To illustrate, when supporting students to reason about number partitions up to five in the 10-frame, the teacher used a narrative involving a watermelon stall with two decks (see Figure 1, left).

The narrative involved a teacher's friend, Doña Esperanza, who sold watermelons at a market. Once students had understood the situation, the teacher asked them to advise Doña Esperanza on arranging a certain number of watermelons on her stall. As students proposed different ways, the teacher kept a record of their suggestions on the board, specifying how many watermelons would be in the top and in the bottom deck (see Figure 1, right).

Figure 1. A watermelon stall and the capture of the record the teacher kept.

On Jesica's suggestion, the learning goals of Phase 3 were pursued in tandem with those of Phase 2. She used the final 20 minutes of sessions 8 to 11 to support her students' development of essential number understandings up to ten, using similar activities to those in Phase 1, but with number words, collections, and written numerals up to 10.

The learning goals of Phase 4 were met over the final 10 sessions of the CDE, and the students came to reason about number patterns and partitions when solving relatively complex arithmetic problems with numbers up to ten (Peña, 2018). In Session 21, the teacher presented the class with a problem about passengers getting on and off a *Tour Bus*. The bus left a park with 4 tourists, made a stop at a museum, and arrived at the final destination with 10 tourists onboard. Students were asked to explain what happened at the museum.

When the teacher first asked the whole class, it was considered obvious that more tourists had boarded the bus. Lupe, asked by the teacher, explained: "Six got on because six are missing for ten". Hernan, when asked whether he understood what Lupe said, responded: "Yes! Six are missing because there were four, and six are ten." Both Lupe and Hernan were amongst the children who, at the beginning of the CDE, showed the least understanding of early number. Their responses to the problem illustrate how, by the end of the CDE, the great majority of the students did not only solve rather complex additive problems correctly, but how they did so by reasoning about number patterns (i.e., the amount 4 is included in the amount 10, the complement of 4 to make 10 is 6), not by counting by ones.

Cycle 2: Modifying the Instructional Sequence for Broader Teacher Use

After finishing her degree, Jesica returned to teaching in a new school and used the instructional sequence she researched, to teach. Initially, she was questioned by her principal and colleagues, as she focussed on small numbers and used whole class activities, instead of "maintaining the challenge" by teaching more complex tasks from the Mexican curriculum that involved larger numbers and working with small groups of students, as was usual in her school. She defended her teaching decisions by referring to her students' assessments (e.g., lack of familiarity with larger numbers), the results of the CDE, and to the research literature.

Her teaching soon became of interest to her principal and supervisor because of what she was achieving with her students. The students were much more eager to participate in mathematics than what was typical in the school, and the parents were very pleased with her teaching. She was asked to give short workshops during the staff meetings, both at the school and the school-cluster levels. Several of her colleagues started to approach her for advice.

At that point, we decided to create a website that contained the materials developed during the CDE, to make these readily available to the interested teachers. It was then that we noticed (based on download data and teachers' questions) how Jesica's colleagues were much more interested in the classroom activities aimed at supporting "essential number understandings" (Table 1, Phases 1, 3), than in the rest of the resource. The teachers readily recognised the importance of their students developing basic counting skills. In contrast, we conjectured, the notion of *patterns and partitions* meant little to them, as did the purpose of related activities.

From their perspective, there were few clear incentives for investing time and effort in learning to pursue the goals of Phases 2 and 4 of the sequence (see Table 1).

We were aware that if teachers were to focus solely on supporting the development of "essential number understandings", their students would not come to solve the *Tour Bus* problems with flexibility when larger numbers were involved, as the only number patterns at their disposal would be those of sequential order (i.e., counting up, and perhaps down, by ones). However, we also knew that in the prevailing teacher culture of Mexican preschool, "problem solving" was a key focus, as it was a goal of Mexican curricula over past decades. Jesica shared that in relation to this goal, there was much frustration amongst her colleagues, because only very few of their students ever became proficient in solving problems.

We thus conjectured that the goals of Phases 2 and 4 of the sequence (see Table 1) would seem much more appealing if preschool teachers recognised them as a means of supporting children to become problem solvers. We introduced modifications aimed at *connecting the learning goals to the problem of teaching* the preschool teachers already faced (see Table 2).

Table 2
The P&P Instructional Sequence Implemented in Cycle 2

Phase	Overarching teaching goal	Specific learning goals
1	Support the development of the *basic number skills* up to five	Master the number word sequence Enumerate with one-to-one correspondence Use fingers to represent numbers Identify the names of written numerals
2	Support the development of *advanced number skills* up to five (problem solving)	Reason about (and subitise) spatial patterns Reason about (and subitise) finger patterns Reason about number partitions in the 10-frame Subitise and reason about spatial patterns in the 10-frame Reason about arithmetic problems
3	Support the development of *advanced number skills* up to ten (problem solving)	*Use basic number skills up to ten (transition)* Reason about (and subitise) finger patterns Subitise and reason about spatial patterns in the 10-frame Reason about number partitions in the arithmetic rack Reason about arithmetic problems

At the crux of these modifications was our intent to convey to preschool teachers that (a) by supporting their students to reach the learning goals of the P&P instructional sequence, they would be providing students with valuable means for problem solving, and (b) while the essential number understandings were necessary for becoming a proficient problem solver, they were not sufficient. The first modification involved renaming "number understandings" to "number skills" to align with the language, in which the teachers made connections to their practice. The second one involved renaming the teaching goals as addressing "basic" vs. "advanced" number skills. This allowed us to both remove the non-transparent "P&P" language and facilitate the image of the advanced number skills as a continuation of the basic skills, which enhanced students' problem solving beyond the activity of counting.

The third modification involved downgrading the Phase 3 (see Table 1) into a transition stage for the following Phase (see Table 2). This were to further support teachers in recognising the P&P goals (now *advanced number skills*) as the key ones in student learning.

By and large, the modifications appeared to be helpful. Jesica reported that the teachers at her school became quite interested in *advanced number skills* and hopeful that these would lead to better problem solving (field journal). Similarly, the two members of the research team who facilitated two 4-hour professional development workshops in Jesica's school, noted that the participating teachers were highly interested in the modified version of the instructional sequence, and that the advanced number skills seemed meaningful to them.

Cycle 3: Highlighting Enjoyment as an Independent Learning Goal in the Sequence

In facilitating the teacher workshops, we became aware that the "fun" activities of Phase 1 were not common in preschool mathematics teaching. We realised then that during the CDE, particularly at the beginning, we had worked to engage the students by making sure that they experienced joy, success, and belonging, regardless of how competent they were. We based the activities on stories and games, and Jesica always focused on conveying to all of her students that they were good at what she was asking them to do.

We thus became aware that this aspect of the teaching during the CDE, which we already took for granted, would not be immediately obvious to, or valued by, the teachers who were interested in the instructional sequence. This led to a further modification, the inclusion of a dedicated Phase 0 (see Table 3), with the main goal of supporting children's willingness to engage in early number activities, and enjoyment of doing so. We developed a set of instructional activities and teaching routines (Lampert et al., 2010) that aimed at supporting teachers in pursuing this initial teaching goal.

Table 3
Addition to the P&P Instructional Sequence Implemented in Cycle 3

Phase	Overarching teaching goal	Specific learning goals
0	Support the development of an *interest in* and a *taste for* counting and numbers	Become interested and show joy when engaging in activities that involve counting or working with numbers

The final version of the instructional sequence has caught the interest of an unexpected number of teachers. In 2020, after Mexican schools closed for COVID-19, we started to collaborate with several teaching organisations and offered intensive online workshops, organised in 2-hour increments over three consecutive days. Although we do not know yet how the attending teachers incorporated the designed resource to their teaching, they valued the experience positively, to a surprising degree. One of the workshops had an attendance of 850 teachers who were present during all three days, even though no external incentives were provided to participate. The Facebook community that we created to keep in touch with the teachers interested in the resource reached 5000 members.

Discussion and Conclusions

In the case of P&P instructional sequence, the research findings that were a product of the initial CDEs addressed the problem of classroom learning. They established that it was to the benefit of learners when the number relationships beyond counting sequence were explored in the classrooms by the means that leveraged quantitative meanings and where students' activity focused on comparing and manipulating quantities that numbers represented. These issues are not addressed by tools such as rainbow numbers, where rainbow—not quantitative pattern—is used to help students memorise numbers with their complements of ten.

Additional design work was essential in making the instructional sequence produced in the CDEs viable for re-sourcing teachers at scale. For us, this included understanding the problems of teaching presently experienced by the teachers, such as the lack of meaning or relevance of some of the design features within their context; the policy-mandated need to support problem solving; and the novelty of legitimacy of cultivation of students' interest in numbers through mathematical activities. We do not claim that the changes in wording and the structure of the resource goals alone made the difference. It was, indeed, critical that the teachers saw, reasonably readily, an associated change in *their* students' engagement, problem solving, and mathematical reasoning. Yet the appropriate wording made the teachers' access to and interest in the ideas presented in the resource possible.

It is at this time not unusual that—like in the Mexican curriculum—the learning outcomes are specified in the form of end-goals (e.g., ability to solve additive word problems in preschool). In absence of resources to suggest otherwise, teachers are encouraged to work *on* the specified end-goals directly (e.g., train students to solve word problems with large numbers). The provision of reasoned classroom journeys towards the curricular end-goals, and the means by which teachers could support such journeys, are needed. It is important to remember that such provision does not seamlessly follow from research on student mathematical learning but requires research into the realities and problems of mathematics teaching.

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